ON THE STRUCTURE OF TURBULENCE IN THE SMALL-SCALE RANGE

(O STRUKTURE TURBULENTNOSTI V OBLASTI MALYKH MASSHTABOV)

PMM Vol.24, No.6, 1960, pp. 1124-1129

G.S. GOLITSYN
(Moscow)

(Received April 26, 1960)

1. The behavior of the turbulence spectrum in the range of high wave numbers, corresponding to the range of viscous dissipation, has been studied in a number of papers. Up to the present, however, this question cannot be regarded as solved. This question is equivalent to the study of the velocity-correlation function for small separations, since the correlation function

$$D_{ij}(r) = \overline{[v_i(M) - v_i(M')]\,[v_j(M) - v_j(M')]}$$

where r is the separation between the neighboring points M and M', is related to the spectral tensor $\Phi_{ij}(\kappa)$ by the relation

$$D_{ij}(r) = 2\iiint\limits_{-\infty}^{\infty}(1 - e^{i\kappa r})\,\Phi_{ij}(\kappa)\,d\kappa \qquad (1)$$

The tensor Φ_{ij} is determined by a unique scalar function - the spectral density $E(\kappa)$:

$$\Phi_{ij}(\kappa) = \frac{E(\kappa)}{2\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) \qquad (2)$$

In connection with the fact that the present statistical theory of turbulence does not provide a closed system of equations for the moments, which in principle would permit a solution of the questions mentioned above, we are forced to have recourse to various kinds of hypotheses, which permit an approximate closure of the system of equations for the moments of Φ_{ij}. A detailed survey of these kinds of hypotheses can be found in [1].

The spectrum of turbulence is known only in the inertial range for values of the wave number much lower than the numbers corresponding to the internal scale of the turbulence. In this range the spectral density

$E(\kappa)$ decreases with increasing wave number, as $\kappa^{-5/3}$. In the range of the internal scale, where the action of viscosity becomes apparent, the intensity of turbulent pulsations rapidly decays, and the spectrum must fall much more quickly. The behavior of the structure function of velocity fluctuations is known not only for scales which are large but also for scales which are small by comparison with the internal scale $r_0 = (\nu^3/\epsilon)^{1/4}$, where ν is the kinematic viscosity, ϵ is the dissipation of energy of turbulent motion per unit mass. If $r \gg r_0$, then the invariant of the structure tensor D_{ij} (its trace) increases with increasing r as $r^{2/3}$. When $r \ll r_0$ the trace $D_{ii} \sim r^2$. The first attempt to determine the structure function throughout the entire range of scale was made by Obukhov [2,3], who made use of the hypothesis of constancy of asymmetry

$$S = D_{lll}(r) / [D_{ll}(r)]^{3/2} = \text{const}, \qquad D_{lll} = \overline{[v_e(M) - v_e(M')]^3} \tag{3}$$

The index l denotes the projection on the direction $\mathbf{r}$. In the inertial range $D_{lll} \sim r$, whilst $D_{ll} \sim r^{2/3}$, so that in fact $S =$ constant. When $r \ll r_0$ the role of the third moments is negligible, and the stated hypothesis cannot greatly distort the results. In fact, a correlation function was obtained with the correct asymptotic forms for large and small distances. However, not every function, even if it has the correct asymptotic behavior, can be a correlation function.

The fundamental requirement of the correlation function lies in the fact that the corresponding spectrum, or more exactly the spectral density, must be positive for all values of the wave number, since it represents the density of kinetic energy per unit mass in the wave space. Generally speaking, it was previously not clear whether Obukhov's correlation function satisfied this requirement. Below in this paper we shall carry out the calculation of the spectrum under the hypothesis of constant asymmetry and shall show that for certain values of the wave number the spectrum has negative values. In conclusion we shall compute correlation functions which correspond to different forms of spectral decay for high wave numbers; from these it will be clear that the character of the spectrum in this range does not influence the form of the correlation function in the corresponding range very strongly.

2. The hypothesis of constant asymmetry allows one to obtain from Kolmogorov's equation [2,3] the equation for the determination of the correlation function

$$\beta' + \left(\frac{4}{3}\beta\right)^{3/2} = x \tag{4}$$

where $\beta(x)$ is the normalized longitudinal correlation function (components of the velocity taken in the direction r), while x is the length taken in the definition of the scale. The normalization here is as follows:

$$r = k_1\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} x, \qquad D_{ll} = k_2 (\nu\varepsilon)^{1/2} \beta(x)$$

$$k_1 = \left(\frac{5}{2}\right)^{1/4} \frac{4}{|S|} \approx \frac{5.035}{|S|}, \qquad k_2 = \left(\frac{2}{45}\right)^{1/4} \frac{4}{|S|} \approx \frac{1.838}{|S|}$$

The value $x = 1$ approximately corresponds to the internal scale of the turbulence. When $x \ll 1$, $\beta \approx 1/2\, x^2$, and when $x \gg 1$, $\beta \approx 3/4\, x^{2/3}$. In the intermediate region Equation (4) is integrated numerically, and the graph of the function $\beta(x)$ is given in [2,3].

Let us suppose for the moment that the turbulence is not only locally homogeneous but in fact purely homogeneous. Then the following expressions hold:

$$R_{ij}(\mathbf{r}) = \iiint_{-\infty}^{\infty} \Phi_{ij}(\varkappa) e^{i\varkappa\mathbf{r}} d\varkappa \tag{5}$$

$$\Phi_{ij}(\varkappa) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} R_{ij}(\mathbf{r}) e^{-i\varkappa\mathbf{r}} d\mathbf{r} \tag{6}$$

where $R_{ij}(r)$ and $\Phi_{ij}(\kappa)$ are the correlation function and the corresponding spectrum. Let us apply the Laplace operator to Expression (5) and then apply the Fourier transform to the result. We then obtain

$$-\varkappa^2 \Phi_{ii}(\varkappa) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \Delta R_{ii}(\mathbf{r}) e^{-i\varkappa\mathbf{r}} d\mathbf{r} \tag{7}$$

Completing the series of transformations, using the conditions of isotropy and continuity, and averaging with respect to angle, we can obtain the following formula for the spectral density:

$$\varkappa E(\varkappa) = -\frac{1}{\pi} \int_0^{\infty} (r^2 f''' + 7rf'' + 8f') \sin(\varkappa r)\, dr \tag{8}$$

where f is the longitudinal correlation function. Let us make use, moreover, of the fact that the derivative of the correlation function f is equal to the derivative of the correlation function D_{ij} multiplied by $(-1/2)$. Since only the derivatives enter into Formula (8), it is true also for the case of locally homogeneous turbulence. Introducing the normalization quoted above, we obtain

$$F(k) \equiv CkE(k) = \int_0^{\infty} (x^2 \beta''' + 7x\beta'' + 8\beta') \sin(kx)\, dx \tag{9}$$

where C is a certain constant determined by the scale factors, while the function $\beta(x)$ is determined by Equation (4). The left-hand side of this expression is a universal function of the normalized wave number k.

Let us present certain results of the numerical integration of Expression (9); the method of integration is given below in an appendix:

k =	0.5	1.0	2.0	5.0	7.5	8.0
$F(k)$ =	6.1241	4.2704	3.6238	0.1725	−0.0034	−0.0424
k =	8.5	9.0	3π	5π	10π	20π
$F(k)$ =	−0.0097	0.0029	0.0124	0.0000	−0.0001	0.0000

The function $F(k)$ first passes through zero when $k = 6.5$. The accuracy of the calculation in the neighborhood of $k = 8$ is estimated as half a unit in the third decimal place (see appendix). Accordingly, the results of the calculation testify to the fact that the spectrum of turbulence for the hypothesis of constant asymmetry becomes negative for certain values of the wave number, although these negative values are relatively small. The rather well-established zeros of $F(k)$ for large wave numbers confirm the theoretical proposition concerning rapid decrease of the spectrum in the range corresponding to the zone of viscous dissipation, and also provide an indication of the sufficient accurary and reliability of the rather complicated system of calculation of the function $F(k)$. In Fig. 1 we present graphs of $F(k)$ on a logarithmic scale - curve 1. Curve 2 is the spectrum smoothed according to Gauss (see (10) below). When $k \ll 1$ the following asymptotic expansion holds good:

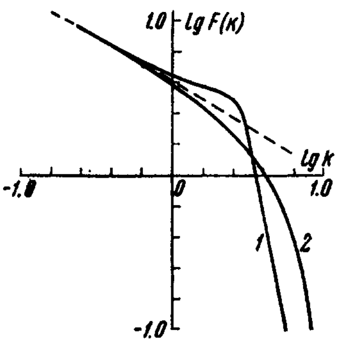

Fig. 1.

$$F(k) = \frac{55\sqrt{3}}{24}\Gamma\left(\frac{5}{3}\right)k^{-2/3} \approx 3.584k^{-2/3}.$$

This asymptote is represented in the figure by a dotted line. The calculated points for $k < 1$ lie well on this line. When $k \gtrsim 1$ the function $F(k)$ rises above the asymptotic line, which indicates that the spectral density $E(k)$ falls more slowly than $k^{-5/3}$. This also does not argue in favour of the hypothesis of constant asymmetry.

Similar calculations were made for the spectrum of temperature fluctuations, corresponding to the correlation function defined by the equation of Iaglom [4], under the hypotheses of constant asymmetry for pulsations of velocity and constancy of the coefficient

$$N = D_{ltt}(r) / D_{tt}(r)\sqrt{D_{ll}(r)}$$

$$D_{ltt} = \overline{[v_l(M) - v_l(M')][T(M) - T(M')]^2}$$

Here D_{ll} and D_{tt} are the correlation functions of velocity and temperature. The spectra so obtained also have negative ranges.

3. Correlation functions for the whole range of scale can be constructed giving a spectrum of turbulence which is positive everywhere. The form of the spectrum is known only for $\kappa \ll 2\pi/r_0$. When $\kappa \gg 2\pi/r_0$ the spectrum rapidly falls to zero, since the intensity of pulsations of velocity on these scales falls rapidly as a result of the action of viscosity. The decrease of the spectrum according to any particular power law arouses great suspicion, since it would denote that certain high derivatives of the velocity do not exist. If we assume that all derivatives of the velocity exist, then it can easily be concluded that the spectrum decreases with increasing κ more quickly than any power of κ. A very simple form of spectrum, having at the same time a certain theoretical basis, is the following:

$$E(\varkappa) = A\varepsilon^{2/3}\varkappa^{-5/3}\exp\left(\frac{-\alpha^2\varkappa^2}{\varkappa_0^2}\right), \qquad \varkappa_0 = 2\pi\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{10}$$

where A and α are certain constants which are specified later. When $\kappa \ll \kappa_0$ Expression (10) gives the required form of spectrum. Since the Navier-Stokes equations are of parabolic type, the pulsations of velocity generated by coarse scale vortices are smoothed in the small scales to the Gaussian function (or close to it), which to a certain extent justifies Expression (10).

For the trace of the correlation tensor D_{ii} we have the spectral expansion (obtained from (1) averaged with respect to angle)

$$D_{ii}(r) = A\varepsilon^{2/3}\int_0^\infty \left(1 - \frac{\sin\varkappa r}{\varkappa r}\right)\varkappa^{-5/3} e^{\frac{-\alpha^2\varkappa^2}{\varkappa_0^2}}\, d\varkappa \tag{11}$$

Let us transform to the dimensionless variables

$$\varkappa r_0 = k \qquad r = x r_0, \qquad D_{ii}(r) = \sqrt{\nu\varepsilon}\, d_{ii}(x)$$

Then

$$d_{ii}(x) = A\int_0^\infty \left(1 - \frac{\sin kx}{kx}\right) k^{-5/3} e^{-\alpha^2 k^2}\, dk \tag{11'}$$

By suitable choice of the parameters A and α, we can cause the main terms of the asymptotic expansions for $x \ll 1$ and $x \gg 1$ to coincide with the corresponding expansions of the trace of the correlation tensor, calculated under the hypothesis of constant asymmetry. Both correlation functions have identical internal scales, defined as the abscissa of the point of intersection of the asymptotes. The integral (11′) is calculated with the help of the known integral [5]

$$\int_0^\infty x^{\lambda-1} e^{-ax^2} \sin(kx)\, dx = \frac{1}{2} a^{-\frac{\lambda+1}{2}} \Gamma\left(\frac{\lambda+1}{2}\right) M\left(\frac{\lambda+1}{2};\ \frac{3}{2};\ -\frac{k^2}{4a}\right) \qquad (\lambda > 0)$$

where M is the degenerate hypergeometric function, and employing the principle of analytic continuation to the values of λ we require. As a result we obtain

$$d_{ii}(x) = \frac{9}{2} A\alpha^{2/3}\Gamma\left(\frac{5}{3}\right)\left[M\left(-\frac{1}{3};\ \frac{3}{2};\ -\frac{x^2}{4\alpha^2}\right) - 1\right] \tag{12}$$

The asymptotic expansions of the trace of the correlation tensor β_{ii} are found with the help of the relations

$$\beta_{ii} = \beta_{ll} + 2\beta_{nn}, \qquad \beta_{nn} = \frac{1}{2x}\frac{d}{dx}(x^2\beta_{ll})$$

where β_{ll} and β_{nn} are the longitudinal and transverse correlation functions. The first few terms of the expansion of β_{ll} when $x \gg 1$ are derived in [3]. The expansion of β_{ll} when $x \ll 1$ can be found by integrating Equation (4) in series. The leading terms of the expansion are the following:

$$\beta_{ii} \approx \frac{11}{4} x^{2/3} \quad \text{when } x \gg 1, \qquad \beta_{ii} \approx \frac{5}{2} x^2 \quad \text{when } x \ll 1$$

Using the known expansions of the degenerate hypergeometric function, we can select the parameters A and α so that the leading terms of the asymptotic expansions coincide.

The correlation function and the corresponding spectrum have been calculated also under the assumption of Tatarskii [6]

$$E(\varkappa) = \begin{cases} B\varepsilon^{2/3}\varkappa^{-5/3} & (\varkappa \leqslant \varkappa_m) \\ 0 & (\varkappa > \varkappa_m) \end{cases} \tag{13}$$

The corresponding trace of the correlation tensor can also be calculated and satisfied by the choice of the parameters B and κ_m necessary to the requirements of the asymptotic expansions.

The results of the calculations for the three different correlation functions are as follows:

$$\begin{aligned} \beta_1 &= 2.5x^2(1 - 0.436x^2 + 0.765x^4), \\ \beta_2 &= 2.5x^2(1 - 1.034x^2 + 1.274x^4), \quad \text{when } x \ll 1 \\ \beta_3 &= 2.5x^2(1 - 1.184x^2 + 1.042x^4), \end{aligned}$$

$$\beta_1 = 2.75x^{2/3}\ (1 - 0.2121x^{-4/3} + O\,(x^{-8/3}))$$
$$\beta_2 = 2.75x^{2/3}\ (1 - 0.5370x^{-2/3} + O\,(x^{-10/3}))\ \text{when}\ x \gg 1 \qquad (14)$$
$$\beta_3 = 2.75x^{2/3}\ (1 - 0.1844x^{-2/3} + 0.0021\sin(7.693x)\,x^{-4/3} + O\,(x^{-7/3}))$$

Here β_1 is the structure function calculated under the hypothesis of constant asymmetry, β_2 is the structure function corresponding to the spectrum smoothed according to Gauss and β_3 corresponds to the spectrum of Tatarskii. These correlation functions are depicted in Fig. 2.

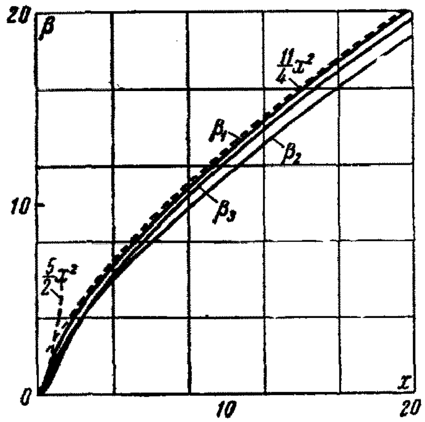

Fig. 2.

When $x \gg 1$ the function β_3 has small oscillations arising as a result of the steep cut-off of the spectrum, but in the figure these oscillations are scarcely visible.

As is clear from the figure, the nature of the decay of the spectrum for high wave numbers does not affect the form of the correlation functions very strongly.

Up to now we have not succeeded experimentally in approaching very closely to the internal scale of turbulence.

However, new methods of investigation, for example, the dispersion of sound in a turbulent atmosphere [7], allow one to hope that the spectrum of turbulence can be directly measured even in the region where the effect of viscosity is beginning to make itself apparent. Experiment will then decide which of the models is closest to reality.

Appendix. Let us describe the calculation of the function

$$F\,(k) = \int_0^\infty (x^2\beta''' + 7x\beta'' + 8\beta')\sin(kx)\,dx \equiv \int_0^\infty f\,(x)\sin(kx)\,dx$$

In principle, knowing the function $\beta(x)$, from Equation (4) we can define also its required derivatives. However, the function $\beta(x)$ is known in the form of a table with a rather small number of values, and the determination even of the first of its derivatives by this method leads to large errors. Therefore, we are led to seek another approach.

For $x \ll 1$ and $x \gg 1$ we have found the asymptotic expansions

$$f(x) = 15x - \frac{68}{3}\sqrt{\frac{2}{3}}\,x^3 + \frac{20}{3}x^5 + O(x^7) \qquad (x \ll 1)$$

$$f(x) = \frac{55}{18}x^{-1/3} + \frac{7}{54}x^{-5/3} + \frac{7}{36}x^{-3} + O(x^{-13/3}) \qquad (x \gg 1)$$

These expansions may be used when $x \leqslant 0.5$ and $x \geqslant 5.0$. For the derivative $\beta'(x) \equiv y(x)$ we obtain from (4) the equation

$$y' = 1 - 2y(x - y)^{1/3}$$

Let us make the substitution: $y = x - v^{3/2}$. Then for $v(x)$ we obtain the simpler equation

$$v' = \frac{4}{3}(x - v^{3/2})$$

which can also be integrated by the method of Runge-Kutta for values of x from 0.5 to 5.0. Relating β'' and β''' to $v(x)$ gives the relations

$$\beta''(x) = 1 - 2v^{1/2}y, \qquad \beta'''(x) = 4vy - 2v^{1/2} - \frac{4}{3}y^2v^{-1/2}$$

As a result

$$f(x) = 8y + 7x(1 - 2v^{1/2}y) + x^2\left(4vy - 2v^{1/2} - \frac{4}{3}y^2v^{-1/2}\right)$$

The accuracy of calculation of this expression in the stated interval is quite satisfactory.

For the calculation of the function $F(k)$ with different values of k, Filon's method was used [8], which is a particular modification of Simpson's method to the case of rapidly oscillating trigonometric functions. This method was used for values of x from 0 to 5, and since for $f(x)$ the asymptotic expression is known it is not difficult to calculate the asymptotic representation for the integral from 5 to ∞. Precise estimates of the error of Filon's method are, generally speaking, unknown. We may imagine, however, that it is determined in just the same way as the error of Simpson's method, i.e. it is equal to 1/15 of the difference between the values of the integral as calculated, and as calculated with twice the interval size. All values of the function $F(k)$ presented in the table were calculated with the interval 0.1. The value of $F(k)$ when $k = 8.0$, calculated with interval 0.2, turns out to be equal to $-$ 0.0211. Therefore we can say that $F(8.0) = -0.0424 \pm 0.0014$.

In conclusion, I express my thanks to A.M. Obukhov for posing the problem and giving advice, and also to L.A. Diky and V.I. Tatarskii for useful advice in the process of carrying out the work.

BIBLIOGRAPHY

1. Batchelor, G.K., *Teoriia odnorodnoi turbulentnosti* (*Theory of Homogeneous Turbulence*). IIL, 1955. (Russian translation of the original, published by Cambridge University Press, 1953).

2. Obukhov, A.M., Lokal'naia struktura atmosfernoi turbulentnosti (Local structure of atmospheric turbulence). *Dokl. Akad. Nauk SSSR* Vol.67, No. 4, 1949.

3. Obukhov, A.M. and Iaglom, A.M., Mikrostruktura turbulentnogo potoka (Microstructure of turbulent flow). *PMM* Vol. 15, No. 1, 1951.

4. Iaglom, A.M., O lokal'noi strukture temperaturnogo polia turbulentnogo potoka (On the local structure of the temperature field of turbulent flow). *Dokl. Akad. Nauk SSSR* Vol. 69, No. 6, 1949.

5. Oberhettinger, F., *Tabellen zur Fourier Transformation*. Springer-Verlag, 1957.

6. Tatarskii, V.I., *Teoriia fliuktuatsionnykh iavlenii pri rasprostranenii voln v turbulentnoi atmosfere* (*Theory of the Form of Fluctuations in the Propagation of Waves in a Turbulent Atmosphere*). *Izd-vo Akad. Nauk SSSR,* 1959.

7. Kallistratova, M.A., Eksperimental'noe issledovanie rasseianiia zvuka na turbulentnosti v atmosfere (Experimental investigation of the dispersion of sound by turbulence in the atmosphere). *Dokl. Akad. Nauk SSSR* Vol. 125, No. 1, 1959.

8. Tranter, C.J., *Integral'nye preobrazovaniia v matematicheskoi fizike* (*Integral Transforms in Mathematical Physics*), Section 5.3. GITTI, 1956.(Russian translation of the original, published by Methuen 1951).

Translated by A.H.A.